SECOND LATIN AMERICAN & CARIBBEAN
UNIVERSITY PHYSICS OLYMPIAD
(LACUPO)
May 11, 2018

Sociedad Cubana de Física

PERSONAL DATA:

Name: ____

Higher education center: ____

Country:___________________**Career:**____________________________**Year:**_____________________

Phone: ____________________________ **E-mail:**_________________________________

ID number:_______________________________

SIGNATURE: __________________________

SCORE: 1:___ , 2:___ , 3:___ , 4:___ , 5:___ TOTAL:________

ON THE SOLUTIONS:

- **Please write the solutions to different problems, in separate pages.**
- **Electronic Calculators are allowed.**

SCORES:

- **The number of points assigned to each problem is written right after the corresponding title. Partial solutions will be assigned a number of points.**

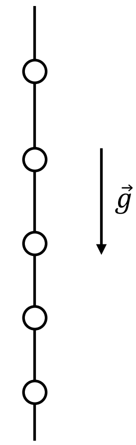
DURATION OF THE EXAM:

- **4½ hours.**

Problem 1: Dynamics of a vertical abacus (20 points)

The figure shows a system made of five small, identical and elastic beads that are allowed to slide along a very long rod placed vertically and resting on the ground. The friction between the rod and the beads can be neglected, and the gravity acceleration along the whole length of the rod is constant and equal to $\boldsymbol{g}$. At the moment $\boldsymbol{t_0}$, different velocities are communicated to each bead, directed either vertically upward or downward.

a) Determine the maximum number of collisions between the beads.



b) Find a relation between the initial speeds of the beads for which, from the reference frame of the ground, the total kinetic energy of the beads reaches the initial value after a certain time $\boldsymbol{t}$. Find an expression for $\boldsymbol{t}$ as a function of $\boldsymbol{g}$ and the initial speeds of the beads.

c) Now assume that the beads were separated by equal distances, $\boldsymbol{d}$, at the beginning of the motion, and the lowest one was a distance $\boldsymbol{H}$ above the ground. Describe one combination for the beads motion resulting in all beads reaching the ground simultaneously after a time $\boldsymbol{t'}$, as a function of $\boldsymbol{H}$, $\boldsymbol{d}$, $\boldsymbol{g}$ and the initial velocities of the beads.

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Problem 2: Does the Evolution of Species violate the Second Principle of Thermodynamics?

(20 points)

a) The DNA of a bacterium contains approximately 5×10^6 base pairs, each base being either AT, TA, GC or CG*. From all the possible configurations, only around 10^{13} correspond to living species. What is the entropy variation needed for a mutation of any configuration to result in a living species?

b) Assume that the evolution of species takes place so fast, that every 100 years each species evolves into another one 1000 times more organized and, hence, 1000 times less probable. Taking into account that there are around 10^{32} living organisms on Earth, estimate the entropy variation in our planet resulting from each second of evolution.

c) In average, the Earth absorbs $P = 1.21 \times 10^{17}\ W$ of solar energy and re-emits it almost entirely to the outer space, so its average temperature is around 288 K. The temperature of the surface of the Sun is of 5778 K and that of the outer space if the temperature of the microwave radiation background, i.e., approximately 2,7 K. How much the entropy changes during one second due to the Sun – Earth – Outer Space energy exchange?

d) In the light of the previous questions, do you think that there is any contradiction between the origin of life, the Evolution of Species and the Second Principle of Thermodynamics? Please explain.

* A=Adenine, C=Cytosine, G= Guanine, T= Thymine

Datum: Use the value $k \approx 1.38 \cdot 10^{-23}\ JK^{-1}$ for Boltzmann's constant.

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Problema 3: Detecting the magnetic monopole (20 points)

Usually, Maxwell equations are written as follows:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (1) \qquad\qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J_e} \qquad (2)$$

$$\vec{\nabla} \cdot \vec{D} = \rho_e \qquad (3) \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0 \qquad (4)$$

where $\vec{E}$, $\vec{H}$, $\vec{D}$, $\vec{B}$, ρ_e y $\vec{J_e}$ are the electric field, the magnetic field, the electric flux density, the magnetic flux density, the electric charge density and the electric current density, respectively.

Some theoretical models suggest the existence of magnetic monopoles (analogous to the electric charge), and predict that each monopole has a magnitude of $4 \times 10^{15}\ Tm^2$. If the theory was correct, Maxwell equations could be "completed" by adding the terms ρ_m and $\vec{J_m}$, representing the magnetic charge density and the magnetic current density, respectively, associated to magnetic monopoles.

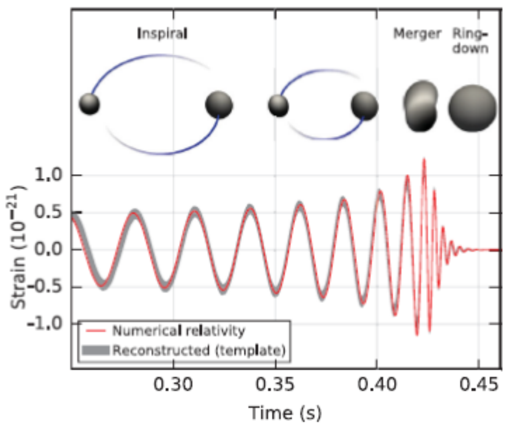
a) Re-write Mawell's equations, assuming the existence of magnetic monopoles.

b) Until today, the existence of magnetic monopoles has not been proven experimentally. However, they have been searched since the beginning of the 1980's using a superconducting ring located inside a shielded environment, where the "conventional" magnetic noise is minimized. Use the Maxwell equations re-written by yourself in (a) to demonstrate that it is possible to detect the eventual passage of a magnetic monopole through the hole of the superconducting ring by measuring the magnetic flux in the hole. Assume that $\vec{E} = 0$ inside the superconducting material, and that $\vec{B}$ is uniform in the ring's orifice.

c) Let us suppose that the superconducting ring has been fabricated from a 1 μm-thick, high critical temperature superconducting film, in such a way that the plane of the orifice is in the plane of the film. What is, approximately, the maximum inner radius of the ring so it is able to detect the passage of one magnetic monopole while the system is immersed into a refrigerant at T ≈ 77 K, considering just the effects of the thermal noise? Assume that $\vec{B}$ is uniform inside the orifice, and zero inside the superconducting material.

Problem 4: Gravitational waves.[1] (20 points)

In 1916, Albert Einstein predicted that accelerated mass emits Gravitational Waves (GW) propagating at the speed of light in vacuum (c). They deform the space as they move, then modifying the dimensions of bodies. These waves were detected by the first time by September 14th, 2015 due to an astrophysical event labeled GW150914, allowing to observe by the first time: i) black holes 30 times more massive than the Sun (whose mass is $M_{\odot}$); ii) a system of two black holes rotating around their common center of mass and iii) the spiral evolution of the latter system until the two black holes fused together. The 2017 Nobel Prize in Physics was granted to R. Weiss, B. C. Barish and K. S. Thorne for their decisive contributions to the creation of the LIGO detector (*Laser Interferometer Gravitational-Waves Observatory*) and the observation of GW using it.



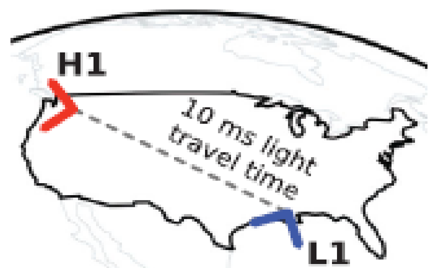


Fig. 1 Left panel: Sketch of the evolution of a binary system of black holes observed in 2015, and the relative deformation of a length during the detection of GW150914 by one of the LIGO observatories. Right panel: Localization of the LIGO detectors that recorded the GW150914 in Livingston (L1) and Hanford (L2).

The evolution of the binary system before their fusion can be approximately described by Newtonian mechanics.

a) If the two black holes are represented by point-like masses m_1, m_2 located at a distance d from each other, show that they rotate around the common center of mass with an angular velocity given by:

$$\omega^2 = \frac{G(m_1 + m_2)}{d^3}$$

where G is the universal gravitational constant.

b) Show that, when the common center of mass is in repose, the mechanical energy of the binary system is given by:

$$E = -\frac{Gm_1 m_2}{2d} = -\frac{G^{2/3} m_1 m_2}{2(m_1 + m_2)^{1/3}} \omega^{2/3}$$

[1] This problem is inspired in the papers: B.P. Abbott et al., *The basic physics of the binary black hole merger GW150914*, Ann. Phys. (Berlin) 529, No. 1-2, 1600209 (2017) and H Mathur, K Brown, A Lowenstein, *An analysis of the LIGO discovery based on Introductory Physics*, American Journal of Physics 85 (9), 676 (2016). Figures were taken from B. P. Abbott et al., PRL, 116, 061102 (2016).

c) As the distance between the black holes decreases, their mechanical energy decreases due to the emission of GW, whose frequency increases in time as $f = \omega/\pi$. Following the theory of general relativity, the power emitted is proportional to the square of the moment of inertia, I as:

$$P = \frac{32G}{5c^5} I^2 \omega^6$$

The way in which f evolves in time is related with the so-called "chirp mass": $\equiv \frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1 + m_2)^{\frac{1}{5}}}$. Assuming that the power emitted as GW's is equal to the loss of mechanical energy of the system, show that the chirp mass is given by:

$$M = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-\frac{8}{3}} f^{-\frac{11}{3}} \frac{df}{dt}\right)^{\frac{3}{5}}$$

Actual measurements of f and its temporal derivative $\frac{df}{dt}$ associated to the event GW150914 allowed to determine a mass of M$\sim 30\ M_{\odot}$.

d) Assume that the black holes have equal masses. Determine their masses as a function of the solar mass, as well as the separation d_0 when the GW frequency reaches its maximum value ($f_{max} = 150\ Hz$).

e) The minimal distance an object can be from a black hole is called Schwarzschild radius: beyond it, even light cannot escape its attraction. Show that the distance calculated in the previous question is bigger than the minimal distance two black holes can be before fusing together.

f) Calculate the energy emitted as GW's during the evolution of the binary system, since the moment they were separated by a distance of d_0. Express it as a function of $M_{\odot}$.

g) How would you explain that the gravitational wave associated to GW150914 took approximately 7 ms to travel from the LIGO observatory in Livingston to that in Hanford, if the distance between the two is of $\sim 3000\ km$?

Use the following values of the constants: $G = 6.67 \cdot 10^{-11}\ {m^3}/{s^2 \cdot kg}$; $c = 2.99 \cdot 10^{8}\ {m}/{s}$; $M_{\odot} = 2{,}00 \cdot 10^{30} kg$.

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Problem 5: The mass of protons and neutrons. (20 points)

Protons and neutrons (nucleons) carry most of the mass in usual matter on Earth and all the known planets. Nucleons are supposed to be constituted by three quarks of two different types: Up (u) and Down (d). Among the six types of quarks recognized by current particle theory, these are the ones with smallest masses. Let us assume that they can be regarded as photons, i.e., as particles with zero-mass, energy $\epsilon = \hbar\omega$ and linear momentum $\vec{p} = \frac{\epsilon}{c^2}\vec{c}$, were $\vec{c}$ is a vector representing the direction of motion with modulus equal to the velocity of light in the vacuum. It is known that, between two quarks, there is a potential energy that increases linearly with the inter-quark distance, $(\vec{x_1} - \vec{x_2})$, according to $V(\vec{x_1} - \vec{x_2}) = k|\vec{x_1} - \vec{x_2}|$, where k is a constant. That potential energy is assumed to "confine" the quarks, so they cannot behave as free particles. Let us further assume that (i) the motion of the three quarks (for example, inside a proton) take place within a plane, (ii) that their energies are identical, and (iii) that the distance between quarks is also identical and constant in time. Finally, let us assume that a quasi-classical, Bohr-like quantization rule holds for the quarks, in such a way that $pL = 2\pi n\hbar$, with $n = 1, 2, 3 \ldots$ where *p* and *L* are the linear momentum and the perimeter of the quark's orbit.

a) Provide an estimate of the proton's mass in GeV by using the information presented above, and the experimental value for the radius of the proton, which is $0.82\ Fermi$ ($1\ Fermi = 10^{-15}\ m$). Compare your estimate with the experimental mass of the proton ($0.938\ GeV$).

b) Evaluate the constant k of the quark confinement potential in $GeV/Fermi$.

c) Using the value of k obtained in (b), calculate the mass of a meson (made of two quarks) under the same assumptions used in the case of the proton.

d) Given the values estimated for the mass of the quarks m_u and m_d, which is of the order of a few MeV, evaluate if the approximation of massless quarks is acceptable. Estimate the velocity of a quark with a mass of $5\ MeV$ with the energy calculated assuming the zero-mass approximation.

Useful data:
$\hbar = 1.054 \times 10^{-34}\ Js$ $\quad c = 2.99 \cdot 10^8$ m/s $\quad 1\ GeV \equiv 1.602 \times 10^{-10}\ J$
$1m \equiv 10^{15} Fermi$ $\quad m_d = 2.5 - 5\ MeV$ $\quad m_u = 1.5 - 3\ MeV$ $\quad m_\pi = 0.136\ GeV$

Note: The term *mass* in this problem corresponds to the term *mass at rest*, following certain textbooks. We have also used the fact that this mass can be expressed in units of energy by applying Einstein's relation between mass and energy.